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$\mathcal{W}$ geometry from Fedosov's deformation quantization

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Abstract

A geometric derivation of W_∞ gravity based on Fedosov's deformation quantization of symplectic manifolds is presented. To lowest order in Planck's constant it agrees with Hull's geometric formulation of classical non-chiral W_∞ gravity. The fundamental object is a $\mathcal{W}$-valued connection one form belonging to the exterior algebra of the Weyl algebra bundle associated with the symplectic manifold. The $\mathcal{W}$-valued analogs of the self-dual Yang–Mills equations, obtained from a zero curvature condition, naturally lead to the Moyal Plebanski equations, furnishing Moyal deformations of self-dual gravitational backgrounds associated with the complexified cotangent space of a two-dimensional Riemann surface. Deformation quantization of W_∞ gravity is retrieved upon the inclusion of all the $\hbar$ terms appearing in the Moyal bracket. Brief comments on non commutative geometry and M(atrix) theory are made.

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1. Introduction

Since the classic work of Bayen et al. [1], Moyal deformation quantization techniques [2–5] are starting to become very relevant in the area of non-commutative geometry, for example (see [21] for a current review). W_∞ algebras, strings, gravity, etc. were very popular candidates to extensions of the ordinary two-dimensional conformal field theory (CFT) description of strings based on Kac-Moody and Virasoro algebras . For an extensive review on higher conformal spin extensions of CFT we refer to the Physics Reports article [8]. Not long ago, the author [14,15] was able to show that non-critical W_∞ strings are devoid of BRST anomalies for target spacetimes of dimension $D = 27$. The supersymmetric case

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yielded $D = 11$ and we suggested that the an anomaly free (super) membrane should support non-critical W_∞ strings in their spectrum.

W_∞ covariance is of crucial importance in the Weyl–Wigner–Groenewold–Moyal quantization process [7] and for this purpose its relevance should be investigated further. The role of W_∞ algebras in Moyal quantization was also investigated by [24]. The authors in [32,33] were able to realize that nonlinear W_∞ algebras could be obtained by Moyal deformations of the classical w_∞ linear ones. Other important connections among membranes and strings were raised in [34].

In this paper we present a very natural framework, in the language of Fedosov's deformation quantization [6], where the original formulation of W_∞ geometry presented by Hull [9,10] can be incorporated in a very simple fashion. In Section 2 we discuss Fedosov's geometry and Moyal SDYM theories in R^4.

In Section 3.1 we shall present an indispensable review of Hull's formulation of W_∞ gravity, while in Section 3.2, we will show how Hull's formulation of W_∞ gravity, written in terms of a single action containing higher derivatives of a single scalar field, is a just the $\hbar^0$ component of a more general action corresponding to the Fedosov deformation quantization of two-dimensional symplectic manifolds. The full action containing the Fedosov's deformation quantization of Hull's action is constructed, to all powers in $\hbar$, at the end of Section 3.2.

In Section 4 we shall follow a different route than the one based on Yang–Mills theories and use the chiral-model approach to self dual gravity. The Fedosov deformation quantization techniques will be applied to such chiral models yielding deformations of the *second* heavenly equations associated with self dual gravity (versus the first heavenly equations). We will also discuss Quantum $\mathcal{W}$-valued Yang–Mills theories in $D = 2$ and their relation to the quantization of Hull's action and W_∞ gravity. Finally, in Section 4.2, the method of induced W_∞ gravity from WZNW models and their suitable Fedosov deformation quantization are discussed at the completion. This last construction is based on deformations of the co-adjoint orbits method of the area-preserving diffs algebra [59], the W_∞ algebra.

In Section 5 we outline some of the many applications that deformation quantization techniques have in the theory of extended objects.

2. Fedosov geometry and Moyal self-dual Yang–Mills

2.1. Fedosov's deformation quantization

In essence, Fedosov's [6,42–46] deformation quantization; i.e. deformations of the Poisson Lie algebra structure on a symplectic manifold, is a generalization of the Bayen et al. [1] and the Weyl, Wigner, Groenewold, Moyal (WWGM) [2–5] deformation quantization on flat phase spaces. Given a symplectic manifold, $\mathcal{M}$ of dimension $2n$, with a symplectic globally defined non-degenerate two-form ω, allows to define a symplectic structure in each tangent space, $T_x\mathcal{M}$ to the point x. The Weyl algebra, $\mathcal{W}_x$ corresponding to the symplectic space, $T_x\mathcal{M}$, is the associative non-commutative algebra over C with a unit element. The elements of the Weyl agebra are defined by a formal power series:

$$a(y) = \sum h^k a_{k,i_1\cdots i_l} y^{i_1}\cdots y^{i_l}, \qquad 2k + l \geq 0. \tag{1}$$

where h is a formal parameter which can be identified with $\hbar$ and the coordinates, $y^1, \cdots, y^{2n} \in T_x\mathcal{M}$ are associated to a tangent vector at the point x. The degrees 1, 2 are prescribed for the variables y, h respectively, so one is summing in Eq. (1) over elements of the algebra of non-negative degree. The non-commutative product on the Weyl algebra, W_x, which determines its associative character, of two elements, $a(y)$, $b(y)$ is defined:

$$a(y) \bullet b(y) = \sum_{k=0}^{\infty} \left(\frac{i\hbar}{2}\right)^k \frac{1}{k!}\omega^{i_1 j_1}\cdots\omega^{i_k j_k}(\partial_{i_1\cdots i_k} a)(\partial_{j_1\cdots j_k} b), \tag{2}$$

where ω^{ij} are the components of the inverse tensor of ω_{ij}, the sympectic two-form.

Having defined the Weyl algebra at a point $x \in \mathcal{M}$ one can defined an algebra bundle structure by taking disjoint unions of the Weyl algebras at each point; i.e. a fiberwise addition. In this fashion a Weyl algebra bundle, $\mathcal{W}$, is constructed as x runs over all the points in $\mathcal{M}$. This allows to build in a set of sections, $\mathcal{E}(\mathcal{W})$, denoted by $a(x, y, \hbar)$ which can be written in a power series:

$$a(x, y, \hbar) = \sum h^k a_{k,i_1\cdots i_l}(x) y^{i_1}\cdots y^{i_l}, \qquad 2k + l \geq 0, \tag{3}$$

where now $a_{k,i_1\cdots i_l}(x)$ are a set of smooth functions defined on the symplectic manifold. Differential forms can be constructed whose elements are $\mathcal{W}$-valued (instead of the customary forms taking values in a Lie algebra). A $\mathcal{W}$-valued q differential form may be written as

$$A(x, y, \hbar) = \sum h^k a_{k;i_1\cdots i_l;j_1\cdots j_q}(x) y^{i_1}\cdots y^{i_l}\,\mathrm{d}x^{j_1}\wedge\cdots\wedge dx^{j_q}, \quad 2k + l \geq 0, \tag{4}$$

and an exterior algebra extension of the Weyl algebra bundle, $\mathcal{E}(\mathcal{W}\otimes\Lambda)$ is obtained by constructing a deformation of the wedge product, $\wedge^{\bullet}$. Differential operators analogous to the exterior derivative and its dual (divergence) are defined in [6] and reviewed in [17,42–46]. We shall not repeat it here.

A torsion-free symplectic connection, ∂, preserving the symplectic structure can also be defined. The symplectic connection does not change the degree of the Weyl algebra bundle, $\partial : \mathcal{E}(\mathcal{W}_p\otimes\Lambda^q) \to \mathcal{E}(\mathcal{W}_p\otimes\Lambda^{q+1})$. Fedosov defined a more general connection, D, the analog of a gauge covariant derivative, acting in an element, a, of the Weyl algebra bundle as

$$Da = \partial a + \frac{1}{i\hbar}[\gamma, a]. \quad \gamma \in \mathcal{E}(\mathcal{W}\otimes\Lambda^1), \tag{5}$$

for γ a globally defined $\mathcal{W}$-valued one form. Since every symplectic manifold can be equipped with an almost complex structure, J, such that the tensor defined by $g(X, Y) = -\omega(JX, Y)$ for all vector fields, X, Y, is a Riemannian metric on $\mathcal{M}$. This yields the curvature R associated with the symplectic connection: $R = 1/4 R_{ijkl} y^i y^j\,\mathrm{d}x^k\wedge\mathrm{d}x^l$. The commutator is defined as: $[a, b] = a \wedge^{\bullet} b - (-1)^{qp} b \wedge^{\bullet} a$, where a, b are respectively $\mathcal{W}$-valued q, p differential forms. The curvature of the connection, D, satisfies the property:

$$D^2 a = \frac{1}{i\hbar}[\Omega, a]. \quad \Omega = R + \partial\gamma + \frac{1}{i\hbar}\gamma^2. \tag{6}$$

A connection D is abelian if and only if, for any section, $a \in \mathcal{E}(\mathcal{W} \otimes \Lambda^1)$, $D^2 a = 0$. Hence from (6) one can infer that the curvature of every abelian connection, Ω, is central (commutes with all the elements a of the Weyl algebra) and is proportional to $\omega_{ij}\, dx^i \wedge dx^j$. Abelian connections are very relevant in the construction of the algebra of quantum observables, $\mathcal{E}(\mathcal{W}_D)$, which is the subalgebra of the Weyl algebra bundle comprised of *flat sections*,

$$a \in \mathcal{E}(\mathcal{W}_D) \Rightarrow D^2 a = 0. \tag{7}$$

these flat sections (with respect the abelian connection) generate a subalgebra of the Weyl algebra bundle called the algebra of quantum observables. It is the analogous of BRST invariant states in string theory.

Finally one may establish the relationship with the ordinary Moyal star product. Fedosov showed that one can assign to a flat section (relative to an abelian connection) $a(x, y, \hbar)$ a central element of the Weyl algebra, $a_o(x, y = 0, \hbar) \in \mathcal{Z}$, and vice versa. The bijection map (which we shall not go into details) and the star product in the central elements of the algebra, $a_o, b_o \in \mathcal{Z}$ are:

$$\sigma : \mathcal{W}_D \to \mathcal{Z}, \quad \sigma(a) = a_o, \quad \sigma^{-1} a_o = a, \quad a_o * b_o = \sigma(\sigma^{-1}(a_o) \bullet \sigma^{-1}(b_o)) = \sigma(a \bullet b), \tag{8}$$

in this way one can construct a star product in the space of central sections, $\mathcal{Z}$, inherited from the non-commutative associative Fedosov fiberwise product of flat sections belonging to the subalgebra of quantum observables. When the symplectic manifold is flat, R^{2n}, the star product reduces then to the ordinary Moyal star product as expected. In this particular case the tangent bundle is trivial, $T(R^{2n}) \sim R^{2n} \times R^{2n}$ and there is no difference between the fiberwise (y^i) Fedosov products and the base space (x^n) Moyal ones. For definitions of the trace of quantum observables in the Weyl algebra bundle, its inner automorphisms (analogous to gauge transformation), symplectic diffs,... we refer to the literature [6,17].

2.2. Moyal self dual Yang–Mills

It has been known for some time to the experts that $4D$ self dual gravity can be obtained from $SU(\infty)$ SDYM, an effective $6D$ theory, by dimensional reduction. We refer to [16] for an extensive list of references. The bosonic and supersymmetric case was studied also by the author [11–13]. Moyal deformations of self dual gravity were proposed by Strachan [25] and rotational Killing symmetry reductions furnished the Moyal quantization of the continuous Toda field [11–13] . Generalized Moyal Nahm and continuous Moyal Toda equations were developed by [36] based on the work of the SDYM equations by Ivanova and Popov [55–57]. W_∞ is a natural symmetry of these integrable theories. Takasaki [26] and Strachan, among many others, have emphasized the importance of higher dimensional integrable theories.

In this section we will write down the main equations of Moyal deformations of SDYM theories in R^4 leaving all the technical details for the reference [16,17]. The basic equations are obtained from a zero curvature condition which allows to gauge two of the fields $A_x, A_y = 0$ and yields for the remaining third equation:

$$\partial_{\tilde{x}} A_{\tilde{y}} - \partial_{\tilde{y}} A_{\tilde{x}} + [A_{\tilde{y}}, A_{\tilde{x}}] = 0. \tag{9}$$

A WWGM quantization of a $SU(N)$ SDYM requires finding a representation of the Lie algebra $su(N)$ in a suitable Hilbert space, where the $\hat{A}_\mu$ are vector-valued operators living in a Hilbert space. A WWGM quantization requires the construction of operators acting in the Hilbert space of square integrable functions on the line, $L^2(R)$, and the use of the symbol WWGM map to define a one-to-one correspondence of self-adjoint operators into real valued smooth functions in the phase space q, p associated with the line. Evenfurther, the WWGM map takes commutators, $1/i\hbar[\hat{A}, \hat{B}]$ into Moyal brackets. In [36] we have shown that in general one should enlarge the phase space with the introduction of q^i, p^i to accomodate for the Moyal deformations of continuum Lie algebras introduced by Saveliev and Vershik [58]. Hence Eq. (9) becomes an effective $6D$ equation after the WWGM quantization:

$$\partial_{\tilde{x}} A_{\tilde{y}} - \partial_{\tilde{y}} A_{\tilde{x}} + \{A_{\tilde{y}}, A_{\tilde{x}}\}_{\text{Moyal}} = 0. \tag{10}$$

where now, $A_\mu(x, y, \tilde{x}, \tilde{y}, q, p, \hbar)$. The last equation admits, in the $\hbar \to 0$ limit, reductions to many of the known integrable equations, like the Plebanski first and second heavenly equations, the Park-Hussain and Grant equations,.... [16]. Furthermore, Eq. (10) may be obtained directly from a Lagrangian [16]. In fact, dynamics of higher spin fields can be encoded in zero curvature constraints. This has been explained in particular by Vasiliev [53,54]. The $6D$ Moyal SDYM equations (10) admit a reduction to $4D$ as follows:

$$A_{\tilde{y}} = \partial_x \Theta = \partial_x \Theta' - \tfrac{1}{2}\tilde{x}, \quad A_{\tilde{x}} = -\partial_y \Theta = -\partial_y \Theta' + \tfrac{1}{2}\tilde{y}, \tag{11}$$

with the $6D$ function Θ :

$$\Theta(x, y, \tilde{x}, \tilde{y}, q, p, \hbar) \equiv \Theta'(x + \tilde{y}, \tilde{x} - y, q, p, \hbar) - \tfrac{1}{2}(x\tilde{x} + y\tilde{y}). \tag{12}$$

Inserting (11) and (12) into (10) yields the $4D$ Moyal first heavenly Plebanski equation:

$$\{\Omega_w, \Omega_{\tilde{w}}\}_{\text{Moyal}} = 1, \quad \Omega(w, \tilde{w}, q, p, \hbar) \equiv \Theta', \qquad w = x + \tilde{y}; \tilde{w} = \tilde{x} - y. \tag{13}$$

Eq. (13) can also be rewritten, due to the fact that the Moyal bracket of the variables $x, y, \tilde{x}, \tilde{y}$ is zero:

$$\{\partial_x \Theta(x, y, \tilde{x}, \tilde{y}, q, p, \hbar), \ \partial_y \Theta(x, y, \tilde{x}, \tilde{y}, q, p, \hbar)\} = 1. \tag{14}$$

The dimensional reduction of the Moyal SDYM theory, from $6D \to 4D$, leading to the Moyal Plebanski equation, can be interpreted as a foliation of the $6D$ space into $4D$ leaves endowed with Moyal deformed (self-dual/anti-self-dual curvatures) hyper-Kahler Ricci flat metrics and parametrized by the coordinates $\tilde{x}$, $\tilde{y}$ of the $6D$ space:

$$\{\partial_x K_{\tilde{x},\tilde{y}}(x, y, q, p, \hbar), \partial_y K_{\tilde{x}\tilde{y}}(x, y, q, p, \hbar)\}_{\text{Moyal}} = 1. \tag{15}$$

where $K_{\tilde{x}\tilde{y}}(x, y, q, p, \hbar)$ is a two-parameter family of Moyal deformed Kahler potentials. This is attained by setting in (14) and (15):

$$\begin{aligned} \partial_y K_{\tilde{x}\tilde{y}}(x, y, q, p, \hbar) &= \partial_y \Theta(\tilde{x}, \tilde{y}|x, y, q, p, \hbar), \\ \partial_x K_{\tilde{x}\tilde{y}}(x, y, q, p, \hbar) &= \partial_x \Theta(\tilde{x}, \tilde{y}|x, y, q, p, \hbar). \end{aligned} \tag{16}$$

Hence, for running values of $\tilde{x} = \tilde{x}_o; \tilde{y} = \tilde{y}_o$ which characterize the foliation, Eqs. (13)–(15) yield a two-parameter family of Moyal heavenly metrics encoded in the Moyal Plebanski potential: $\Omega(w, \tilde{w}, q, p, \hbar)$. The foliation is represented by the two-parameter family of four-dimensional non-commutative manifolds, $X^4_{\tilde{x},\tilde{y}}(x, y, q, p, \hbar)$, since the star product of two functions in phase space is non-commutative. The connections between non-commutative geometry, Matrix models [18,19,21,22] and String theory [20] is now being developed by a large number of authors. We apologize for excluding many relevant references. The relevance of self dual gravity in connection with $N = 2$ strings was initiated by [27] and pursued by many others [39,40]. Important remarks about M(atrix) models and $N = 2$ strings have appeared in [38] . The role of self-dual gravity and W_∞ algebras was initially worked out, among others, by [28–31]. This completes this short review about Moyal SDYM in R^4.

3. $\mathcal{W}$-Geometry from Fedosov

3.1. Hull's formulation of W_∞ gravity

We consider it important to present a short review of Hull's formulation of W_∞ gravity prior to presenting the main results of this work. Sometime ago, Hull [9,10] with great insight, presented a geometric formulation of W_∞ geometry as a gauge theory of the group of symplectic diffeomorphisms of the cotangent bundle of a two-dimensional Riemann surface, $\text{Diff}_o(T^*\mathcal{N})$. The infinite set of symmetric gauge tensor fields are $\tilde{h}^{\mu_1\cdots\mu_n}_{(n)}$, $n = 2, 3, \ldots \infty$; $\mu = 0, 1$. These transform as densities under $\mathcal{W}$ gauge tranformations:

$$\begin{aligned} \delta\tilde{h}^{\mu_1\cdots\mu_s}_{(s)} = \sum_{m,n} \delta_{m+n,s+2} \Big[&(m-1)\lambda^{(\mu_1\mu_2\cdots}_{(m)} \partial_\nu \tilde{h}^{\cdots\mu_s)\nu}_{(n)} - (n-1)\tilde{h}^{\nu(\mu_1\mu_2\cdots}_{(n)} \partial_\nu \lambda^{\cdots\mu_s)}_{(m)} \\ &+ \frac{(m-1)(n-1)}{p-1} \partial_\nu \left\{ \lambda^{\nu(\mu_1\mu_2\cdots}_{(m)} \tilde{h}^{\cdots\mu_s)}_{(n)} - \tilde{h}^{\nu(\mu_1\mu_2\cdots}_{(n)} \lambda^{\cdots\mu_s)}_{(m)} \right\} \Big]. \end{aligned} \tag{17}$$

s labels the conformal spin $2, 3, 4, \ldots, \infty$ of the gauge fields whose physical components have helicity $\pm s$. From the generalized notion of a scalar line element:

$$\mathrm{d}s = (g_{\mu_1\cdots\mu_n}\, \mathrm{d}x^{\mu_1} \cdots \mathrm{d}x^{\mu_n})^{(1/n)}$$

Hull proposed an action of the form:

$$S = \int d^2x \tilde{F}(x, y), \quad \tilde{F}(x, y) = \sum_{n=2}^{\infty} \frac{1}{n} \tilde{h}_{(n)}^{\mu_1 \cdots \mu_n} y_{\mu_1} \cdots y_{\mu_n}, \tag{18}$$

where the function $\tilde{F}(x, y)$ is a co-metric $\mathcal{W}$-density in $d = 2$ instead of a $\mathcal{W}$-scalar. The action represents the integrated generalized world interval along a section of the bundle $T^*\mathcal{N}$, where the fiber coordinates, y_μ, when restricted to a section, Σ, can be interpreted as the gradients of the matter fields $y_\mu|_\Sigma = y_\mu(x) = \partial_\mu \phi(x)$. Hull used one and only one bosonic scalar field, $\phi(x)$, living in the two-dimensional world sheet. Later we will see how to include a set of matter fields, $\phi^i(x)$ representing the embedding spacetime coordinates of the string worldsheet. The nonlinear transformation property of the field $\phi(x^\mu)$ is

$$\delta\phi = \Lambda(x^\mu, y_\mu) = \sum_{n=2}^{\infty} \lambda_{(n)}^{\mu_1 \cdots \mu_n}(x^\mu) y_{\mu_1} \cdots y_{\mu_n}. \tag{19}$$

However, the formulation based on the infinite number of fields $\tilde{h}_{(s)}^{\mu_1 \cdots \mu_s}$ was redundant (the action was reducible) in the sense that there are more gauge fields than are needed. For example, $\tilde{h}^{\mu\nu}$ has three independent components for only two gauge symmetries. Hull proved that a gauge invariant (invariant under the transformations given by (17) and (19) constraint could be consistently imposed on all the gauge fields in such a way that one could recast the action solely in terms of a set of unconstrained fields, $h_{(s)}^{\mu_1 \cdots \mu_s}$ with their traces removed at the linearized level after exploiting the $\mathcal{W}$-Weyl conformal invariance. Thus, at the linearized level with respect to a flat $\eta_{\mu\nu}$ two-dim metric and to lowest order in the gauge fields one has

$$\tilde{h}_{(s)}^{\mu_1 \cdots \mu_s} = [h_{(s)}^{\mu_1 \cdots \mu_s} - \text{traces}] + O(h^2) \cdots \qquad \delta\tilde{h}_{(s)}^{\mu_1 \cdots \mu_s} = \partial^{(\mu_1} \lambda_{(s)}^{\mu_2 \cdots \mu_s)} + \cdots ;$$
$$\lambda_{(s)}^{\mu_1 \cdots \mu_{s-1}} = [k_{(s)}^{\mu_1 \cdots \mu_{s-1}} - \text{traces}] + O(k^2) \cdots \tag{20}$$

The gauge-invariant background independent master constraint that generates all the constraints on the gauge fields, upon expansion in powers of y_μ, and which allows to recast the action solely in terms of the unconstrained fields is

$$\det[\tilde{G}^{\mu\nu}(x^\mu, y_\mu)] = 1, \quad \tilde{G}^{\mu\nu}(x^\mu, y_\mu) = \frac{\partial^2 \tilde{F}(x^\mu, y_\mu)}{\partial y_\mu \partial y_\nu}. \tag{21}$$

For 2+2 signature one has −1 for the determinant instead in the r.h.s. of (21). The constraint (21) can be solved in a particular way by recurring to the $\mathcal{W}$-Weyl conformal invariance which allows to gauge away all the traces of the unconstrained fields, leaving only traceless fields in the action with helicities $\pm s$.

The geometrical significance of the constraint (21) is the following. If one sets $y_\mu = z_\mu + \bar{z}_\mu$, where z_μ are the complex coordinates on R^4, $\mu = 0, 1$, allows to view $(x^\mu, z_\mu, \bar{z}_\mu)$ as the coordinates for the bundle C $T_x^*\mathcal{N}$, the complexification of the cotangent space $T_x^*\mathcal{N}$ at a point $x \in \mathcal{N}$. The co-metric function is then reinterpreted as:

$$\tilde{F}(x^\mu, y_\mu) = K_x(z, \bar{z}) = \tilde{F}(x^\mu, z_\mu + \bar{z}_\mu). \tag{22}$$

with $K_x(z, \bar{z})$ the Kahler potential depending on the combination $z_\mu + \bar{z}_\mu$ which is tantamount of a Killing symmetry reduction condition, the metric does not depend on the two imaginary components of z_μ:

$$\frac{\partial^2 K_x(z,\bar{z})}{\partial z_\mu \partial \bar{z}_\nu} = \frac{\partial^2 \tilde{F}(x^\mu, y_\mu)}{\partial y_\mu \partial y_\nu}. \tag{23}$$

The gauge invariant (under the transformations given by (17) and (19) master constraint (21) was equivalent to the Monge-Ampere equation (Plebanski equation) for a Ricci flat, hyper-Kahler manifold associated with the compexified cotangent space at each point x^μ, of the original two-dimensional surface, world-sheet, $\mathcal{N} : CT^*_x\mathcal{N} \sim C^2$. This implies that for each x^μ, the corresponding curvature tensor is either self-dual or anti-self-dual. Summarizing, for each, $x^\mu \in \mathcal{N}$, $\tilde{F}(x^\mu, y_\mu = z_\mu + \bar{z}_\mu) = K_x(z, \bar{z})$ is the Kahler potential for a hyper-Kahler metric on R^4 with two commuting tri-holomorphic Killing vectors. Hence, $K_x(z, \bar{z})$ furnishes a two-parameter family of metrics labeled by the points $x^\mu \in \mathcal{N}$ and a bundle over $\mathcal{N}$ is obtained whose fibers at each point are isomorphic to C^2 and equipped with a half-flat metric with two Killing vectors.

When $d = 2$ it is possible to construct invariant actions under a subgroup of the $\mathcal{W}$ transformations (symplectic) if, and only if, a $\mathcal{W}$ scalar-density exists, $\tilde{F}(x^\mu, y_\mu)$ so that the action is $\mathcal{W}$-invariant up to surface terms. In $d > 2$ no such density exists. The reason that the action was solely invariant under a subgroup of $\mathrm{Diff}_o(T^*\mathcal{N})$ was due to the fact that a constraint on the gauge parameters $\lambda^{\mu_1\cdots\mu_s}_{(s)}$ had to be imposed as well because under $\mathcal{W}$ transformations given by Eqs. (17) and (19), the action behaves like

$$\delta S = \int \mathrm{d}^2x\, \partial_\mu(\Omega^\mu + X) = 0 \Rightarrow X = 0. \tag{24}$$

To lowest order in $\Lambda(x^\mu, y_\mu)$ the constraint $X = 0$ reads:

$$\det\left[\frac{\partial^2(\tilde{F} + \Lambda)(x^\mu, y_\mu)}{\partial y_\mu \partial y_\nu}\right] = 1. \tag{25}$$

for $2 + 2$ signature one has -1 on the r.h.s. of (25). The above constraint represents *infinitesimal* deformations of the hyper-Kahler geometry with two-Killing vectors, for a deformed Kahler potential $\tilde{F} \to \tilde{F} + \Lambda$. The constraint $X = 0$ on the gauge parameters is not fully symplectic-diffs invariant like Eq. (21) was, it is only invariant under a subgroup of the symplectic-diffs [9,10]. In the next section we will present a straighforward derivation of all these constraints based on Moyal deformations of SDYM.

Summarizing, invariant actions under a subgroup of the symplectic-diffs can be constructed in terms unconstrained gauge fields, $h^{\mu_1\cdots\mu_s}_{(s)}$, and transformation laws with gauge parameters, $k^{\mu_1\cdots\mu_s}_{(s)}$. Their traces can be removed by means of exploiting the $\mathcal{W}$-Weyl conformal invariance. These traceless (irreducible) fields and parameters appear in the invariant action and transformation laws *nonlinearly* in the form of $\tilde{h}^{\mu_1\cdots\mu_s}_{(s)}$, and $\lambda^{\mu_1\cdots\mu_s}_{(s)}$. The symmetry algebra of non-chiral W_∞ gravity is a subalgebra of the $\mathrm{Diff}_o(T^*\mathcal{N})$.

For references on the twistor transform, reductions to W_N Geometry and its connection to Strominger's special geometry [37], finite versus infinitesimal transformations, etc... we refer to Hull's original work [9,10]. Now we are ready to recast the constraints (21) and (25) in the language of Fedosov–Moyal quantization.

3.2. $\mathcal{W}$-Geometry from Moyal–Fedosov

In the following sections we shall derive the two main results of this work. From the last Sections 2.2 and 3.1, we can see that Hull's construction of W_∞ geometry fits very naturally with the Moyal self dual gravitational equations (14) and (15) (after a Killing symmetry reduction) . The $\hbar = 0$ limit in Eqs. (14) and (15) of the Moyal brackets turns into Poisson brackets and the latter, in turn, can be formulated as a simple determinant:

$$\{\partial_x K_{\tilde{x},\tilde{y}}(x, y, q, p, \hbar = 0), \partial_y K_{\tilde{x},\tilde{y}}(x, y, q, p, \hbar = 0)\}_{\text{Poisson}} = 1$$
$$\Rightarrow \det\left[\frac{\partial^2 K_{\tilde{x},\tilde{y}}(x, y, q, p, \hbar = 0)}{\partial\xi^i\partial\bar{\xi}^j}\right] = 1. \tag{26}$$

where $\xi^1;\xi^2$ plus complex conjugates conjugates $\bar{\xi}^1;\bar{\xi}^2$ are suitable functions of the x, y, q, p variables. Two Killing symmetry reductions must be subsequently performed in such a way that the x, y, q, p dependence appears solely in the combinations $y^i = \xi^i + \bar{\xi}^i, i = 1, 2$. In this fashion one can make contact with the two-Killing symmetry reductions imposed by Hull [9,10]. The four variables x, y, q, p admit a natural interpretation in terms of the $z_\mu, \bar{z}_\mu$ variables which described the C^2 fibers of the complexified cotangent space of the two-dim surface, $\mathcal{N}$ at a given point $x^\mu \in \mathcal{N}$. The $\hbar = 0$ limit of the Moyal heavenly equations associated with the two-parameter family of leaves foliating the $6D$ space, after a further Killing symmetry reduction $y^i = \xi^i + \bar{\xi}^i$, admit a direct correspondence with the constraints (21) and (23) present in Hull's formulation:

$$K_{\tilde{x},\tilde{y}}(x, y, q, p, \hbar=0) = K_{\tilde{x},\tilde{y}}(y^i = \xi^i + \bar{\xi}^i, \hbar = 0) \leftrightarrow K_{x^\mu}(z_\mu, \bar{z}_\mu)$$
$$= K_{x^\mu}(y_\mu = z_\mu + \bar{z}_\mu) = \tilde{F}(x^\mu, y_\mu). \tag{27}$$

Since we were able to identify $K_{\tilde{x},\tilde{y}}(x, y, q, p, \hbar)$ with a two-parameter family of Moyal-Kahler potentials, $\Theta(\tilde{x}, \tilde{y}|x, y, q, p, \hbar)$, it follows that Hull's co-metric density function can naturally be embedded into Θ by performing the *double* infinite summation typical of Fedosov's Geometry and recurring to the two Killing symmetry reductions:

$$\Theta(\tilde{x}, \tilde{y}|x, y, q, p, \hbar) = \sum_{2n+l\geq 0} \hbar^n \Theta_{n;i_1,\cdots i_l} \, (\xi + \bar{\xi})^{i_1} \cdots (\xi + \bar{\xi})^{i_l}. \tag{28}$$

The coefficients belong to a one parameter family of smooth tensorial functions of the $\tilde{x}, \tilde{y}$ variables (parametrized by the integer n):

$$\Theta_{n;i_1,\cdots i_l}(\tilde{x}, \tilde{y}). \tag{29}$$

Hull's co-metric density corresponds solely to the $\hbar^0$ terms:

$$\mathcal{F}(x^\mu, y_\mu) \leftrightarrow \sum_{l\geq 0}(\xi + \bar{\xi})^{i_1} \cdots (\xi + \bar{\xi})^{i_l}\Theta_{0;i_1,\ldots i_l}(\tilde{x}, \tilde{y}). \tag{30}$$

while the higher order $\hbar$ corrections implement in one scoop the Moyal deformation quantization of non-chiral W_∞ gravity. The series in (30) must start with $l = 2$ in order to match the expression for the co-metric density. A truncation to zero of the first two terms in the series is necessary. Futhermore, we have an expansion in contravariant vectors versus Hull's expansion in terms of covariant vectors. Indices are raised and lowered using the two-dim metric. Eq. (30) may be reinterpreted as the zeroth-order terms (in $\hbar$) associated with a Fedosov deformation quantization of the symplectic two-dim manifold with coordinates $\tilde{x}, \tilde{y}$. The tangent vector at the each point should have for coordinates $y^i = \xi^i + \bar{\xi}^i$. This is relevant to the chiral model approach to self-dual gravity [17] where instead of starting with a direct Yang–Mills formulation one writes down the Fedosov deformations of WZNW actions.

If the aforementioned interpretation is adopted, Hull's action will be the integration of the $\hbar^0$ component of Θ along a section of the cotangent bundle of the two-dim manifold. Notice that this is not the same as taking the Fedosov trace of the zero form, Θ, belonging to the exterior algebra of the Weyl algebra bundle associated with the two-dim symplectic manifold (Riemann surface). Secondly, the W_∞ transformations of the higher spin fields and the matter fields in Eqs. (17) and (19) must not be confused with the Fedosov's non-commutative fiberwise product algebra of the y^i coordinates. Although the W_∞ algebra can be identified with Moyal deformations of the classical area-preserving diffs [32,33]. The $\hbar$ corrections to the Hull action are then

$$S = \int \mathrm{d}^2x \sum_{n=0}^{\infty} \sum_{l}^{\infty} \hbar^n \Theta_n^{\mu_1,\mu_2\ldots\mu_l}(x^\mu)\partial_{\mu_1}\phi(x^\mu)\ldots\partial_{\mu_l}\phi(x^\mu) \tag{31}$$

with the condition $2n + l \geq 0$ on the double summation meaning that one is summing over terms of positive degree of the Weyl algebra, $x^\mu = x^0, x^1$.

Concluding: Eq. (31) is one of the main results of this work. Hull'action corresponds solely to the $\hbar^0$ terms and to those values of l such that $l \geq 2$. The remaining terms of the action, in powers of $\hbar$, are the corresponding ones due to the Fedosov deformation quantization of non-chiral W_∞ gravity.

The integral in Eq. (31) is taken along a *section*, Σ, of the cotangent bundle of the two-dimensional manifold, $T^*\mathcal{N}$, where $\mathcal{N}$ is parametrized by the coordinates x^0, x^1. The fiber coordinates y^μ, restricted to the section Σ, can be identified with the gradients of the matter field : $y_\mu|_\Sigma = y_\mu(x) = \partial_\mu\phi(x)$ with $\mu = 0, 1$.

We emphasize once again that the action Eq. (31) does not have the form of Fedosov's trace of the zero form of the algebra, Θ. Fedosov's trace is relevant in the study of $\mathcal{W}$-valued YM theories in $D = 4$ and its dimensional reduction to $D = 2$. It is in this context that one can evaluate the Fedosov's trace of the zero form Θ. A *zero* form may be seen as the *prepotential* of a $\mathcal{W}$-valued YM field: $\mathcal{A}$ such that $\partial\Theta = \mathcal{A}$; where ∂ is the torsion-free symplectic connection taking a $\mathcal{W}$-valued q differential form into a $\mathcal{W}$-valued $q + 1$ differential form. We should recall that the torsion free symplectic connection $\partial \neq d$ where $\mathrm{d} \equiv \mathrm{d}x^i \wedge \partial/\partial x^i$.

The Yang–Mills like curvature is, $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge^\bullet \mathcal{A}$ and from this, a Yang–Mills like action $\mathrm{tr}\mathcal{F} \bullet \mathcal{F}$ can be constructed using Fedosov's trace. Dimensional reductions from

$D = 4$ to $D = 2$ will yield more general actions than those of Hull. In particular, quantum deformations of generalized YM theories where the gauge fields take values in the Weyl algebra bundle associated with symplectic manifolds instead of the Lie algebra of a group G. At the end of Section 4.1 we will write down the action for these $\mathcal{W}$-valued YM theories in $D = 2$.

It is important not to confuse the $\mathcal{W}$-valued YM theories living in a particular symplectic manifold, with the $\mathcal{W}$-valued chiral models. The latter models involve a set of fields living in a given manifold but taking values in the Weyl algebra bundle associated with another manifold, a curved symplectic space, whose coordinates are q, p and whose fibers are y^1, y^2, for example. Whereas, the former $\mathcal{W}$-valued YM theories are genuine theories living on the original symplectic manifold. Therefore, the $\mathrm{tr}\mathcal{F} \bullet \mathcal{F}$ Yang–Mills like actions on $D = 2$, for example, are not to be confused with a $\mathcal{W}$-valued $2D$ chiral model involving the $\mathcal{W}$-valued maps from one $2D$-manifold to the Weyl algebra bundle of another symplectic manifold. This will become clear in the next example.

To finalize this section it is worthy of mentioning that the other constraint (25) that broke the full symplectic-diffs invariance down to a subgroup can also be understood within the framework of Moyal SDYM. The analogy of gauge transformations of the Moyal YM potentials reads:

$$A_M \to A_M + \partial_M \lambda + \{A_M, \lambda\}_{\mathrm{Moyal}}, \quad F_{MN} \to \{F_{MN}, \lambda\}_{\mathrm{Moyal}}, \tag{32a}$$

under gauge transformations the zero curvature (SDYM) conditions are preserved and hence the Moyal Plebanski equations are naturally gauge invariant exactly like it happened to the constraint (21). However, one can see that $\Theta \to \Theta + \Lambda$ is not always a symmetry of the Moyal SDYM theory. Under shifts in Λ the Moyal YM potentials do not transform as (31) but instead:

$$A_{\tilde{x}} = -\partial_y \Theta \Rightarrow A_{\tilde{x}} \to A_{\tilde{x}} - \partial_y \Lambda. \tag{32b}$$

Eq. (32a),(32b) and (32c) does not represent a true gauge transformation of the $A_{\tilde{x}}$ Moyal YM potential due to the gauge non-covariant $-\partial_y \Lambda$ piece. One would only have true gauge covariance if, and only if, Λ, λ satisfy the property:

$$-\partial_y \Lambda = -\partial_y \Lambda - \{A_y, \Lambda\}_{\mathrm{Moyal}} = \partial_{\tilde{x}} \lambda + \{A_{\tilde{x}}, \lambda\}_{\mathrm{Moyal}} = \delta A_{\tilde{x}}, \tag{32c}$$

which implies the two conditions:

$$\{A_y, \Lambda\}_{\mathrm{Moyal}} = 0, \quad -D_y \Lambda = D_{\tilde{x}} \lambda. \tag{33}$$

this is a very restricted condition on the gauge parameter λ and the shift parameter Λ. It is not surprising that the symplectic-diffs invariance is not fully preserved under shifts in Λ of the Moyal Kahler potential Θ. We recall from [16] that A_y could be gauged to zero due to the zero curvature condition that furnishes the Moyal SDYM equations in R^4. Hence, the gauge $A_y = 0$ obeys one of the conditions. However, there still remains the other condition restricting the parameters λ, Λ in a highly nonlinear manner in terms of the remaining Moyal YM potentials. Similar considerations apply for the other $A_{\tilde{y}}$ Moyal YM potential.

4. Chiral models approach to self-dual gravity and WZNW actions

4.1. Chiral models and the second heavenly equation

Instead of the foliation picture presented above of the $6D$ space into a family of Ricci flat $4D$ leaves, based on the relation of the Moyal SDYM equations, the *first* heavenly equations and Hull's formulation of W_∞ gravity; i.e the constraints (21) on the Kahler potential, we shall pursue another route. This requires now a formulation based on the chiral model approaches to self dual gravity and their relation with the *second* heavenly equations and to extend the construction of Section 2.2 to the case that the phase space manifold parametrized by q, p is no longer flat. The fact that Hull's construction is based on non-chiral W_∞ gravity is no obstacle to follow the chiral model approach to self dual gravity. At the end one must take the direct sum over the chiral/anti-chiral sectors : $W_\infty \oplus \bar{W}_\infty$.

Once again, we wish to reiterate that one must not confuse the chiral model approach discuss in this section with those of $\mathcal{W}$-valued YM theories. Nevertheless there is a relation between the two cases. To this end we shall present below how a chiral model may be related with YM theories in $D = 2$.

The chiral model we are studying is the following : Let us start with a four-dim manifold, $\mathcal{M}^4$, representing the tangent bundle $T\mathcal{N}$ of a $2D$ manifold $\mathcal{N}$ parametrized by the x, y coordinates, whose fibers have for coordinates the $\tilde{x}, \tilde{y}$ variables. We collectively label the coordinates of $\mathcal{M}^4$ by x^μ, for $\mu = 1, 2, 3, 4$. The gauge fields, $\tilde{A}_\mu$, live on $\mathcal{M}^4$ and take values in the Weyl algebra-bundle $\mathcal{W}$ constructed over a curved two-dim phase space manifold, Γ, with coordinates q, p and whose tangent-space fibers are y^1, y^2. The gauge fields can be represented as: $\tilde{A}_\mu(x^\mu|q, p, y^1, y^2; \hbar)$. The zeroth-element is the central section of the algebra $\mathcal{E}(\mathcal{W}_D)$ obtained from the projection: $\sigma(\tilde{A}_\mu) = \tilde{A}_\mu(x^\mu|q, p, 0, 0; \hbar) = A_\mu(x^\mu|q, p; \hbar)$. Fedosov gave the relation which allows one to reconstruct the full $\tilde{A}_\mu$ field from the knowledge of the central section: A_μ. It resembles an expansion in terms of Riemannian normal coordinates,

$$\begin{aligned} &\tilde{A}_\mu(x^\mu|q, p, y^1, y^2; \hbar) \\ &\quad = A_\mu(x^\mu|q, p; \hbar) + y^i \partial_i A_\mu(x^\mu|q, p; \hbar) + \tfrac{1}{6} y^i y^j y^k \partial_i \partial_j \partial_k A_\mu(x^\mu|q, p; \hbar) \\ &\quad\quad - \tfrac{1}{24} y^i y^j y^k R_{ijkl} \omega^{lm} \partial_m A_\mu(x^\mu|q, p; \hbar) + \ldots \end{aligned} \tag{34}$$

where the derivatives ∂_i are taken w.r.t the q, p coordinates associated with the symplectic manifold Γ: $\partial_1 = \partial_q$; $\partial_2 = \partial_p$. ω^{lm} is the inverse of the symplectic form on Γ and R_{ijkl} its curvature on Γ associated with the symplectic connection.

The $\mathcal{W}$-valued Yang–Mills in $D = 2$ can be obtained from the chiral model by identifying the four-dim manifold $\mathcal{M}^4 \equiv \mathcal{TN}$ with the tangent bundle of the two-dim curved phase space, $\mathcal{T}\Gamma$, such as

$$(x, y) \leftrightarrow (q, p); \quad (\tilde{x}, \tilde{y}) \leftrightarrow (y^1, y^2). \tag{35a}$$

and after imposing the conditions on the gauge fields:

$$\tilde{A}_{\tilde{x}}(x^\mu|q, p, y^1, y^2; \hbar) = 0. \quad \tilde{A}_{\tilde{y}}(x^\mu|q, p, y^1, y^2; \hbar) = 0. \tag{35b}$$

after these identifications and conditions are imposed, which is tantamount of a *double* dimensional reduction from an effective $8D$ theory down to an effective $4D$ one, we will have a $\mathcal{W}$-valued Yang–Mills field $\mathcal{A}$ living in a $D = 2$ manifold, parametrized by x, y, and whose tangent space fibers are y^1, y^2. Such $\mathcal{W}$-valued Yang–Mills differential one-form $\mathcal{A}$ living in $D = 2$ is:

$$\mathcal{A}(x, y; y^1, y^2; \hbar) \equiv \mathcal{A}_x\, \mathrm{d}x + \mathcal{A}_y\, \mathrm{d}y \equiv \sum_{k,l} \hbar^k A_{k;i_1,i_2,\ldots i_l;j}(x, y) y^{i_1} y^{i_2} \ldots y^{i_l}\, \mathrm{d}x^j. \quad (36)$$

The components $A_{k;i_1,i_2,\ldots,i_l;j}(x, y)$ of (36) can be read-off directly from the r.h.s. of Eq. (34), after expanding in powers of $\hbar$ each term, collecting exponents in $\hbar$ and imposing the reduction conditions Eqs. (35a) and (35b).

Another way to see how the dimensional reductions to a final $4D$ theory emerges is by starting directly from Eq. (10). Such equation can be obtained from a Lagrangian in the Moyal case by starting with the usual $4D$ self-dual Yang–Mills theory. Eq. (10) represents the deformations of the six-dimensional version of the *second* heavenly equation. The main difference in the curved phase space case is that one is required to use the non-trivial definition of Fedosov's trace and to take Fedosov's fiberwise products instead of Moyal ones.

In the special *flat* phase space limit, $\Gamma \to R^2$, one recovers naturally the action of [16] which reproduces Eq. (10) as its equations of motion. The trace becomes then an ordinary integration w.r.t. the q, p coordinates and the original four-dim action will turn into an effective $6D$ one as a result of taking the Fedosov trace:

$$S_{\mathrm{SDYM}} = \int \mathrm{d}^4x \int \mathrm{d}q\, \mathrm{d}p \left[-\frac{1}{3}\Theta_o * \{\partial_x\Theta_o, \partial_y\Theta_o\} + \frac{1}{2}(\partial_x\Theta_o) * (\partial_{\tilde{x}}\Theta_o) + \frac{1}{2}(\partial_y\Theta_o) * (\partial_{\tilde{y}}\Theta_o) \right]. \quad (37)$$

where $x, y, \tilde{x}, \tilde{y}$ are the coordinates of $\mathcal{M}^4$. The Moyal star product is taken w.r.t. the q, p coordinates of the now flat phase space, $\Gamma = R^2$. The quantity $\Theta_o(x^\mu|q, p, \hbar)$ is a scalar field living on $\mathcal{M}^4$ and taking values in the space of *central* sections of the $\mathcal{E}(\mathcal{W}_D)$ algebra associated with the symplectic manifold $\Gamma = R^2$.

The action (37) yields the equations of motion for the Moyal deformations of the $6D$ version of the *second* heavenly equation. In the classical $\hbar = 0$ limit, the Moyal brackets turn into Poisson ones as usual. A further dimensional reduction of the action given in Eq. (37) from $6D$ to $4D$ of the type : $\partial_x = \partial_{\tilde{x}}$ and $\partial_y = \partial_{\tilde{y}}$ furnishes the desired final $4D$ theory, which in the $\hbar = 0$ limit, retrieves the Park-Hussain second heavenly equations associated with the chiral models approach to $4D$ self dual gravity. Once again, we see that $4D$ self dual gravity is an essential geometrical ingredient in these constructions.

In the more general case that $\hbar \neq 0$, an integration of Eq. (37) w.r.t. the q, p coordinates, accompanied by the dimensional reduction from $\mathcal{M}^4$ to two-dimensions, allows once again to make contact with the Moyal deformations of Hull's two-dimensional action (18) (derived originally directly from Eq. (10)).

Both views should in principle be equivalent since both lead to $4D$ self dual gravity. One is expressed in terms of the *first* heavenly form and the other in terms of the *second* heavenly one. A Darboux transformation relates the first heavenly equation with the second heavenly one (the chiral models).

The last construction Eq. (37) above relied on the fact that the phase space Γ was flat. The main issue, now, is to construct an action S for the more general case that the phase space $\Gamma(q, p)$, is *curved*, and whose flat space limit, renders Eq. (37) for the self dual sector. For this is necessary to introduce, explicitly, the definition of Fedosov's trace. In particular, the $\mathcal{W}$-valued YM theory action in a curved $D = 2$ symplectic manifold is

$$S_{\rm YM} = \int \mathrm{d}^2x \,\mathrm{tr}(\mathcal{F} \bullet \mathcal{F}), \quad \mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge^{\bullet} \mathcal{A}, \tag{38}$$

where $\mathcal{A}$ is given by Eq. (36). In the flat phase space limit, Fedosov's trace in Eq. (38) yields:

$$S_{\rm YM} = \int \mathrm{d}^2x \int \mathrm{d}q \,\mathrm{d}p \,\sigma(\mathcal{F} \bullet \mathcal{F}) = \int \mathrm{d}^2x \int \mathrm{d}q \,\mathrm{d}p \, F_o * F_o. \tag{39}$$

the self dual sector of the action (39) is related to the action of Eq. (37), written in terms of the central section $\Theta_o(x^{\mu}|q, p; \hbar)$, after the dimensional reduction $\partial_x = \partial_{\tilde{x}}$, $\partial_y = \partial_{\tilde{y}}$ is taken in (37).

Concluding, we have presented three different actions, Eqs. (31),(37) and (38), all of them related to quantum Moyal–Fedosov deformations of self dual Yang–Mills, self dual gravity and W_∞ geometry.

The action of Eq. (31) is the Fedosov deformation quantization of Hull's action (18) related to the *first* heavenly equation. Eq. (37) is the action furnishing the Moyal deformations of the *second* heavenly equation associated with the chiral model approach to self dual gravity. Finally, Eq. (38) is the $\mathcal{W}$-valued YM theory action in a curved $D = 2$ manifold. The $\mathcal{W}$-valued analogs of the self dual YM equations will provide generalizations of Hull's construction for arbitrary *curved* two-dim symplectic manifolds.

4.2. The WZNW models and induced W_∞ gravity

The co-orbit geometric induced action for W_∞ gravity [59], associated with the infinite dimensional Lie algebra of deformations of the differential operators of the circle, DOP(S^1), is nothing but the anomalous effective WZNW action for $D = 2$ matter fields coupled to a chiral W_∞ gravity background:

$$S[g] = -\int \mathrm{d}t \,\mathrm{d}x \,\mathrm{Res}_{\xi}(U_0 \mathrm{o} g^{-1} \mathrm{o} \partial_t g) + \frac{c}{4\pi} \int \mathrm{d}t \,\mathrm{d}x \quad \mathrm{Res}_{\xi}[[ln\xi, g] \mathrm{o}\, g^{-1} \,\mathrm{o} \partial_t g \,\mathrm{o}\, g^{-1}$$
$$-\frac{1}{2}\mathrm{d}^{-1}([ln\xi, \mathrm{d}g \,\mathrm{o}\, g^{-1}] \wedge \mathrm{d}g \,\mathrm{o}\, g^{-1})]. \tag{40}$$

The physical meaning of the first term is that of a coupling of the chiral W_∞ WZ field $g = g(\xi, x; t)$ to a chiral W_∞ gravity "background" represented by the point (U_o, c) of the co-adjoint orbit of the dual space $\mathcal{G}^*$ of the centrally extended infinite dimensional Lie algebra $\mathcal{G}$ of the algebra of DOP (S^1). The equivalence of the centrally extended DOP (S^1) infinite dimensional Lie algebra and that of W_∞ was demonstrated in [32,33].

The variable ξ is the one which appears in the correspondence between pseudo-differential operators and symbols : $X(\xi, x) = \sum_k \xi^k X_k(x) \leftrightarrow \sum_k X_k(x)(\partial_x)^k$; i.e ξ represents a momentum-like variable. The coefficients in the ξ expansion of the symbol of the product $X \circ Y$ are given by the infinite series : $\sum_k (X \circ Y)_k(x)\xi^k$. The residue is obtained by the coefficient of ξ^{-1}.

The commutator $[X, Y] = X \circ Y - Y \circ X$. The second integral in (40) contains the integrated anomaly and c is the central charge. d^{-1} is the inverse of the exterior derivative operator acting on the group G. The integrals of (40) are over a one-dimensional curve on the phase space, or coadjoint orbit $\mathcal{O}(U_o, c)$ of the W_∞, with time evolution parameter t. Along the curve the exterior covariant derivative becomes $\mathrm{d} = \mathrm{d}t\,\partial_t$.

The Fedosov extension of the action in (40) for more general phase spaces and their one-dimensional co-adjoint orbits associated with the Fedosov deformations of the algebra of the area-preserving diffs is an interesting project. This will involve an extension of the chiral WZ field $\tilde{g} = \tilde{g}(\xi, x; y^1, y^2, t; \hbar)$ and a modification of the action (40) where the product of symbols is replaced by Fedosov products $\bullet$. For an exposition of the Fedosov quantization of semisimple co-adjoint orbits see Astashkevisch [42–46]. The induced geometric actions in this case will be generalizations of the work of Nisimov and Pacheva. Further analysis of the two main views presented in this work : foliations of $6D$ spaces into families of Ricci flat $4D$ leaves versus reductions of chiral models and the construction of induced geometric W_∞ gravity actions from WZNW models [28,29,59] and its Fedosov deformation quantization will appear elsewhere.

5. Conclusion

We have shown that Hull's formulation of non-chiral W_∞ geometry fits in very naturally inside a larger picture : Moyal–Fedosov geometry. Hull's constraints (21) and (25) have a natural interpretation in the Fedosov–Moyal quantization program. Furthermore, the inclusion of all the powers in $\hbar$, given by the action of Eq. (31), implements in a straightforward fashion the Fedosov's deformation quantization of non-chiral W_∞ gravity.

In general one must consider a Yang–Mills like formulation of $\mathcal{W}$-geometry based on a $\mathcal{W}$-valued connection one-form belonging to the exterior algebra of the Weyl-algebra bundle associated with the symplectic manifold. Such Yang–Mills like action was given in Eq. (38). The chiral model approach to self-dual gravity and WZNW models were also discussed. It is an important project to construct, if possible, generalizations of the geometric induced W_∞ gravity actions [59] based on Fedosov quantization of WZNW models.

Some of the advantages of this formulation of $\mathcal{W}$-geometry are the following;

- The incorporation of many bosonic fields ϕ^i, $i = 1, 2, 3, \ldots, D$ is straightforward in Fedosov's Geometry : the coefficients in the expansion (1) are matrix valued; i.e. the coefficients take values in the bundle Hom (E, E) where E is a vector bundle over $\mathcal{M}$. For more details on this see [6]. The scalars ϕ^i represent the embedding coordinates of the world sheet in a target spacetime background of dimension D. Since these fields are reinterpreted as matrix valued sections with a non-commutative fiberwise Fedosov product it is clear why the embedding coordinates of the string world sheet inherit a non-commutative product structure!

- Quantization of p-branes [50] should be achieved using deformation quantization methods for higher dimensional generalizations of symplectic geometry [47]: the so-called Nambu–Poisson Hamiltonian Mechanics. Deformation quantization of Poisson manifolds has also been discussed by [48,49].
- We hope that the Moyal–Fedosov deformation quantization approach to $\mathcal{W}$ Geometry will provide many new insights into the non-perturbative structure of string theory. In particular the role of W_∞ strings [14,15]. Paraphrasing Fairlie [23]: Moyal stands for "M"; today we advocate that M-theory "stands upside down" for $\mathcal{W}$. Important objections why W geometry is still far from being understood have been recently raised by [52].
- It has been shown in [35] that $4D$ conformal field theory can naturally be formulated in real four-folds endowed with an integrable quaternionic structure and a $4D$ extension of $2D$ CFT on Riemann surfaces was constructed. Quaternionic (Fueter) analyticity played the role of $2D$ holomorphic analyticity. It is warranted to explore further the connections between the self duality and $\mathcal{W}$-Weyl conformal invariance properties of W_∞ geometry and the quaternionic geometry of $4D$ conformal field theory. $4D$ generalizations of the $2D$ WZNW models were studied by [41]. Earlier work in that direction was provided by Park [30]. The fact that quaternions are non-commutative points in the right direction. The non-associative character of octonions suggests that $8D$ non-associative geometries are no longer speculative figments of the imagination but should also come to play an important role in physics.
- The view advocated here of $\mathcal{W}$ geometry as flat foliations in higher dimensions may have an important relation with Zois [51] proposal for a non-perturbative Lagrangian of M theory in $11D$ in terms of characteristic classes of flat foliations (although in in odd dimensions) .
- A forthcoming project involves to write down Yang–Mills types of action characterizing the higher spin field dynamics. In particular to establish the connection to Vasiliev's work [53,54]: higher-spin gauge theories in four, three and two-dimensions and interactions of matter fields based on deformed oscillator algebras have been studied by Vasiliev and others . A reformulation of the dynamical equations of motion, called "the unfolded formulation", in a form of a zero curvature condition and a covariant-constancy condition imposed on an infinite collection of zero-forms allowed Vasiliev to describe all spacetime derivatives of all the dynamical fields and reconstruct these fields by analyticity in some neighborhood of a fixed point. There are many similarities with the work of [53,54] and ours: a zero curvature condition is imposed; a star product also appears in order to describe the nonlinear dynamics; a deformed oscillator algebra realizing the universal enveloping algebra of symplectic groups is essential...

What is required then is to integrate Vasiliev's formulation in the Moyal–Fedosov program.

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